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Inverse updating and downdating for weighted linear least squares using M-invariant reflections[1]

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Abstract

A new method for the weighted linear least squares problem $\min_x \|M^{-1/2}(b - Ax)\|_2$ is presented by introducing a row M-invariant matrix (i.e., $QMQ^T = M$). Our purpose in this paper is to introduce new row M-invariant and row hyperbolic M-invariant reflections. We then show how these row M-invariant reflections can be used to design efficient sliding-date-window recursive weighted linear least squares covariance algorithms, which are based upon rank-k modifications to the inverse like-Cholesky factor R^{-1} of the covariance matrix. The algorithms are rich in matrix–matrix BLAS-3 computations. We also provide computational experiments indicating the numerical stability of the methods. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the weighted linear least squares problem [1,7–9]

$$\min_{x \in \mathbb{R}^n} (b - Ax)^T M^{-1} (b - Ax), \tag{1}$$

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where

$$A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^{m} \text{ and rank}(A) = n, \text{ and } M = \mathrm{diag}(\mu_1, \ldots, \mu_m), \mu_i > 0.$$

It is assumed that $m \geqslant n$. An equivalent formulation of Eq. (1) is

$$\begin{pmatrix} M & A \\ A^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{2}$$

where $\lambda \in \mathbb{R}^{m}, x \in \mathbb{R}^{n}$, see Ref. [9] for more details.

Since rank$(A) = n$ and M is positive definite (2) has a unique solution.

Assume that $Q \in \mathbb{R}^{m\times m}$ and $M \in \mathbb{R}^{m\times m}$, then Q is said to be M-invariant if it is nonsingular and $QMQ^{\mathrm{T}} = M$.

In a pioneering paper by Gulliksson and Wedin [2], it was shown how Eq. (2) could be solved by using the column M-invariant matrices. In this paper we introduce new row M-invariant and row hyperbolic M-invariant matrices, that play the same role as row householder reflections [3] and M-invariant rotations [2].

We will develop efficient algorithms for the recursive weighted least squares problem of the sliding-window type (see Refs. [5,6]).

The outline of this paper is as follows. In Section 2 we introduce the new row M-invariant methods. In Sections 3 and 4 we show how these matrices can be used to efficiently modify weighted least-squares solutions when observations are added to and/or deleted from the linear system. In Section 5 some computational experiments and some concluding remarks are provided.

2. Row M-invariant methods

In this section we introduce a row M-invariant method which is a rank-1 modification to the identity matrix eliminating all elements in a row of a matrix.

2.1. Row M-invariant reflections

Lemma 2.1 [2]. *Assume that $Q = I - 2cd^{\mathrm{T}}, d^{\mathrm{T}}Md > 0$, where Q is M-invariant with M nonsingular. Then*

$$Q = I - 2Mdd^{\mathrm{T}}/d^{\mathrm{T}}Md, \quad \text{with } Q^2 = I,$$

i.e., Q is a reflector. The matrix Q is called an M-invariant reflection.

Theorem 2.2. *Let B be an $(n+1)\times n$ matrix*

$$B=\begin{pmatrix} b^{\mathrm{T}} \\ D \end{pmatrix}\begin{matrix} \}1 \\ \}n, \end{matrix} \tag{3}$$

where D is nonsingular, and $M=\mathrm{diag}(\mu_1,\dots,\mu_{n+1})$ with $\mu_i>0$. Then there is an M-invariant reflection Q such that

$$QB=\begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}\begin{matrix} \}1 \\ \}n. \end{matrix} \tag{4}$$

Proof. Let

$$c=\begin{pmatrix} c_1 \\ \tilde{c} \end{pmatrix}\begin{matrix} \}1 \\ \}n, \end{matrix} \qquad d=\begin{pmatrix} d_1 \\ \tilde{d} \end{pmatrix}\begin{matrix} \}1 \\ \}n. \end{matrix}$$

We will construct $Q=I-2cd^{\mathrm{T}}$ such that Eq. (4) is satisfied. Then we obtain the relation

$$\begin{pmatrix} b^{\mathrm{T}} \\ D \end{pmatrix}-2\begin{pmatrix} c_1d_1b^{\mathrm{T}}+c_1\tilde{d}^{\mathrm{T}}D \\ \tilde{c}d_1b^{\mathrm{T}}+\tilde{c}\tilde{d}^{\mathrm{T}}D \end{pmatrix}=\begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}. \tag{5}$$

giving $D^{\mathrm{T}}\tilde{d}=\mu b$ where $\mu=(1-2c_1d_1)/2c_1$.

By Lemma 2.1, we have $c=Md/d^{\mathrm{T}}Md$ and $c_1=\mu_1d_1/d^{\mathrm{T}}Md$ that inserted in the expression for μ gives

$$2\mu\mu_1d_1+2\mu_1d_1^2-d^{\mathrm{T}}Md=0. \tag{6}$$

By expanding $d^{\mathrm{T}}Md$ and using $\tilde{d}=\mu D^{-\mathrm{T}}b$ we have

$$d^{\mathrm{T}}Md=d_1^2\mu_1+\mu^2(D^{-\mathrm{T}}b)^{\mathrm{T}}M_1(D^{-\mathrm{T}}b)$$

enabling us to rewrite Eq. (6) as

$$\mu_1d_1^2+2\mu\mu_1d_1-\mu(D^{-\mathrm{T}}b)^{\mathrm{T}}M_1(D^{-\mathrm{T}}b)=0.$$

Solving this equation for d_1 we get

$$d_1=-\mu\pm\sqrt{\mu^2+\frac{1}{\mu_1}\tilde{d}^{\mathrm{T}}M_1\tilde{d}}.$$

By choosing μ, we obtain d and $Q=I-2Mdd^{\mathrm{T}}/d^{\mathrm{T}}Md$.

In order to avoid rounding errors we choose the negative sige in the sonare root in d_1.

We may choose $\mu=1/\|b\|_2$ and if $\|b\|_2=0$, we set $Q=I$. □

We have the follwing algorithm for determining the M-invariant reflection.

Algorithm 1 (RowMR).

INPUT: An n-vector b, a nonsingular $n \times n$-matrix D, and the weight matrix $M = \mathrm{diag}(\mu_1, \ldots, \mu_n, \mu_{n+1})$.
if $\|b\|_2 = 0$ **then**
$d = 0$, $Q = I$.
else
Put $\mu = 1/\|b\|_2$, and solve $D^T\tilde{d} = \mu b$.
Compute $d_1 = -\mu - \sqrt{\mu^2 + \frac{1}{\mu_1}\tilde{d}^T M_1 \tilde{d}}$. Let $d^T = (d_1, \; \tilde{d}^T)$.
end
OUTPUT: $Q = I - 2Mdd^T/d^T Md$ having the property that the first row of QB is all zeros.

Algorithm RowMR will have good numerical properties as long as $\tilde{d}$ is solved by a numerically stable method.

In Section 3 we consider the problem of annihilating r rows. This is easily done by applying a sequence of M-invariant reflections described above as $Q = Q_r Q_{r-1} \cdots Q_1$.

2.2. Row hyperbolic M-invariant reflections

Let $\Phi = \mathrm{diag}(\pm 1)$, and assume that $Q \in \mathbb{R}^{m \times m}$ and $M \in \mathbb{R}^{m \times m}$, then Q is said to be hyperbolic M-invariant if it is nonsingular and $QM\Phi Q^T = M\Phi$ (see Ref. [10]).

Lemma 2.3. *Assume that* $Q = \Phi - 2cd^T . d^T M\Phi d \neq 0$, *where* $\Phi = \mathrm{diag}(\pm 1)$, *and Q is hyperbolic M-invariant with M nonsingular. Then*

$$Q = \Phi - 2Mdd^T/d^T M\Phi d, \quad \text{with} \;\; Q\Phi Q = \Phi.$$

i.e., Q is a hyperbolic reflector. We call Q a hyperbolic M-invariant reflection.

Theorem 2.4. *Let B be an* $(n+1) \times n$ *matrix of the form*

$$B = \begin{pmatrix} b^T \\ D \end{pmatrix},$$

where D is nonsingular, and $M = \mathrm{diag}(\mu_1, \ldots, \mu_{n+1})$ *with* $\mu_i > 0$. *Assume that* $D^T M_1^{-1} D - 1/\mu_1 bb^T > 0$ *then there is a hyperbolic M-invariant reflection* $Q = \Phi - 2cd^T$, *where* $\Phi = \mathrm{diag}(-1, 1, \ldots, 1)$, *such that*

$$QB = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}, \tag{7}$$

where $\tilde{D} \in \mathbb{R}^{n \times n}$.

Proof. Let

$$c = \begin{pmatrix} c_1 \\ \tilde{c} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ \tilde{d} \end{pmatrix},$$

where $c_1, d_1 \in \mathbb{R}, \tilde{c}, \tilde{d} \in \mathbb{R}^n$.

From Eq. (7) we get

$$QB = \Phi B - 2cd^{\mathrm{T}}B = \begin{pmatrix} -b^{\mathrm{T}} - 2(c_1 d_1 b^{\mathrm{T}} + c_1 \tilde{d} D) \\ D - 2(\tilde{c} d_1 b^{\mathrm{T}} + \tilde{c}\tilde{d}^{\mathrm{T}} D) \end{pmatrix}$$

and $D^{\mathrm{T}}\tilde{d} = \mu b$, where

$$\mu = \frac{-1 - 2c_1 d_1}{2c_1}. \tag{8}$$

By Lemma 2.4, we have $c = Md/d^{\mathrm{T}}M\Phi d$, and $c_1 = \mu_1 d_1/d^{\mathrm{T}}M\Phi d$ which, inserted in Eq. (8), gives

$$2\mu_1 d_1^2 + 2\mu_1 \mu d_1 + d^{\mathrm{T}}M\Phi d = 0. \tag{9}$$

By expanding

$$d^{\mathrm{T}}M\Phi d = -d_1^2\mu_1 + \mu^2(D^{-\mathrm{T}}b)^{\mathrm{T}}M_1(D^{-\mathrm{T}}b),$$

where $M_1 = \mathrm{diag}(\mu_2, \ldots, \mu_{n+1})$, we can rewrite Eq. (9) as

$$\mu_1 d_1^2 + 2\mu_1\mu d_1 + \mu^2(D^{-\mathrm{T}}b)^{\mathrm{T}}M_1(D^{-\mathrm{T}}b) = 0. \tag{10}$$

Solving for d_1 gives

$$d_1 = \frac{-2\mu_1\mu \pm \sqrt{\Delta}}{2\mu_1},$$

where we have used the assumption $D^{\mathrm{T}}M_1^{-1}D - (1/\mu_1)bb^{\mathrm{T}} \geqslant 0$ giving $\mu_1 - b^{\mathrm{T}}D^{-1}M_1D^{-\mathrm{T}}b \geqslant 0$ and $\Delta = 4\mu_1\mu^2(\mu_1 - (D^{-\mathrm{T}}b)^{\mathrm{T}}M_1(D^{-\mathrm{T}}b)) \geqslant 0$.

By choosing μ we attain d_1, d and $Q = \Phi - 2Mdd^{\mathrm{T}}/d^{\mathrm{T}}M\Phi d$. □

We arrive at the following algorithm for determining a hyperbolic M-invariant reflection of the kind described above.

Algorithm 2 (RowHMR).

INPUT: An n-vector b, a nonsingular $n \times n$-matrix D, and $M = \mathrm{diag}(\mu_1, \ldots, \mu_n, \mu_{n+1})$.

if $\|b\|_2 = 0$, **then**

$d = 0$; $Q = \Phi$.

else

Let $\mu = 1/\|b\|_2$ and solve $D^{\mathrm{T}}\tilde{d} = \mu b$.

Compute $d_1 = -\mu - \sqrt{\mu^2 - \frac{1}{\mu_1}\tilde{d}^{\mathrm{T}} M_1 \tilde{d}}$.
Let $d^{\mathrm{T}} = (d_1, \tilde{d}^{\mathrm{T}})$.
end
OUTPUT: $Q = \Phi - 2Mdd^{\mathrm{T}}/d^{\mathrm{T}}M\Phi d$ that has the property that the first row of QB is all zeros.

3. Inverse updating

In this section we will solve the updating problem by M-invariant matrix methods.

Consider the weighted linear least squares problem below

$$\min_{w \in \mathbb{R}^n} \|M_1^{-(1/2)}(s - Xw)\|_2, \tag{11}$$

where $M_1 = \mathrm{diag}(\mu_1, \ldots, \mu_m)$, $\mu_i > 0$ and X is an $m \times n$ matrix with $\mathrm{rank}(X) = n$.

Let $X = QR_1$, where Q is an M-invariant matrix with columns,

$$R_1 = \begin{pmatrix} R \\ 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

and R is an $n \times n$ upper triangular matrix. Then the solution to (11) is given by $w = (R^{-1} \quad 0)Q^{-1}s$.

Suppose k new observations $(Y^{\mathrm{T}} \quad u)$, where $Y^{\mathrm{T}} \in \mathbb{R}^{k \times n}$, and $u \in \mathbb{R}^k$, be added to the dating defining the weighted linear least squares problem (11). We then show how the solution w to (11) can be updated to the solution $\tilde{w}$ to

$$\min_{\tilde{w}} \left\| M^{-(1/2)} \left(\begin{pmatrix} s \\ u \end{pmatrix} - \begin{pmatrix} X \\ Y^{\mathrm{T}} \end{pmatrix} \tilde{w} \right) \right\|_2, \tag{12}$$

where $M = \mathrm{diag}(M_1, M_2)$.

Lemma 3.1. *Let*

$$Q \begin{pmatrix} R \\ Y^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}, \tag{13}$$

where Q is an M-invariant matrix, $R, \tilde{R}$ is upper triangular, then

$$Q^{-\mathrm{T}} \begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix}, \tag{14}$$

where $E^{\mathrm{T}} \in \mathbb{R}^{k \times n}$.

Proof. Let

$$\left(Q^{-\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix}\right)^{\mathrm{T}} = (U \quad E),$$

then Eq. (13) gives

$$I = (U \quad E)\begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} = U\tilde{R},$$

and $U = \tilde{R}^{-1}$. □

Lemma 3.2. *Assume that $\tilde{V} = -R^{-\mathrm{T}}Y$, $\tilde{R}$ given in Eq. (13), Q is M-invariant, and $Q = Q_n \cdots Q_1$, where Q_i are M-invariant reflection with $Q_i^2 = I$, such that*

$$Q^{-\mathrm{T}}\begin{pmatrix} \tilde{V} \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}, \tag{15}$$

where I_k is the $k \times k$ identity matrix and $\tilde{D}$ is a $k \times k$ matrix. Then

$$Q\begin{pmatrix} R \\ Y^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} U \\ 0 \end{pmatrix}.$$

If U is upper triangular, then $U = \tilde{R}$, and

$$Q^{-\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix}.$$

Proof. Since Q is M-invariant, then $Q^{-\mathrm{T}}$ is M^{-1}-invariant. We choose row M^{-1}-invariant matrix $Q^{-\mathrm{T}}$, such that

$$Q^{-\mathrm{T}}\begin{pmatrix} \tilde{V} \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix},$$

where $\tilde{D}$ is nonsingular.

For $Q^{-1}Q = I$ and the definition of $\tilde{V}$, we obtain

$$Q\begin{pmatrix} R \\ Y^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} U \\ \tilde{Y}^{\mathrm{T}} \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \quad \text{and} \quad Q^{-1} = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix}.$$

We get

$$\tilde{Q}_{11}^{\mathrm{T}}(-R^{-\mathrm{T}}Y) + \tilde{Q}_{21}^{\mathrm{T}} = 0, \quad \tilde{Q}_{21} - Y^{\mathrm{T}}R^{-1}\tilde{Q}_{11} = 0,$$

$$Q_{21}\tilde{Q}_{11} + Q_{22}Y^{\mathrm{T}}R^{-1}\tilde{Q}^{11} = 0.$$

Since $Q^{-\mathrm{T}}$ is row M^{-1}-invariant matrix, $\tilde{Q}_{11}$ is nonsingular. We obtain

$$Q_{21} + Q_{22}Y^{\mathrm{T}}R^{-1} = 0 \quad \text{and} \quad \tilde{Y} = Q_{21}R + Q_{22}Y^{\mathrm{T}} = 0.$$

Hence

$$Q\begin{pmatrix} R \\ Y^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} U \\ 0 \end{pmatrix}.$$

If U is upper triangular for

$$Q\begin{pmatrix} R \\ Y^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix},$$

then it is easy from Lemma 3.1 to see

$$Q^{-\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix}. \quad \square$$

Theorem 3.3. *Let Q satisfy the same assumptions as in Lemma* 3.2, *if w is the solution to* (11), *then the solution to* (12) *is given by*

$$\tilde{w} = w - E\tilde{D}^{-\mathrm{T}}(u - Y^{\mathrm{T}}w), \tag{16}$$

with E and $\tilde{D}$ given in Lemmas 3.1 *and* 3.2.

Proof. Let

$$QX = \begin{pmatrix} R \\ 0 \end{pmatrix}, \quad Qs = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

where $s_1 \in \mathbb{R}^n, s_2 \in \mathbb{R}^{m-n}$, Q is M_1-invariant matrix.

Then Eq. (12) can be rewritten as

$$\begin{aligned}
&\min \left\| M^{-1/2}\left(\begin{pmatrix} s \\ u \end{pmatrix} - \begin{pmatrix} X \\ Y^{\mathrm{T}} \end{pmatrix}\tilde{w}\right)\right\|_2 \\
&= \min \left\| M^{-1/2}\left(\begin{pmatrix} s_1 \\ s_2 \\ u \end{pmatrix} - \begin{pmatrix} R \\ 0 \\ Y^{\mathrm{T}} \end{pmatrix}\tilde{w}\right)\right\|_2,
\end{aligned} \tag{17}$$

which is equivalent to

$$\begin{pmatrix} M & \begin{pmatrix} X \\ Y^{\mathrm{T}} \end{pmatrix} \\ \begin{pmatrix} X \\ Y^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} s \\ u \end{pmatrix} \\ 0 \end{pmatrix}. \tag{18}$$

Hence we have

$$\begin{pmatrix} M & \begin{pmatrix} R \\ 0 \\ Y^{\mathrm{T}} \end{pmatrix} \\ \begin{pmatrix} R \\ 0 \\ Y^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} Q_1^{-\mathrm{T}}\lambda \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ u \end{pmatrix} \\ 0 \end{pmatrix}, \tag{19}$$

where

$$Q_1 = \mathrm{diag}(Q,\ I).$$

Since $w = R^{-1}s_1$, moreover let $\tilde{V} = -R^{-\mathrm{T}}Y$. Consequently, from Lemma 3.2, there exist M-invariant matrix $\tilde{Q}$, such that

$$\tilde{Q}^{-\mathrm{T}}\begin{pmatrix} \tilde{V} \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix} \quad \text{and} \quad \tilde{Q}^{-\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-\mathrm{T}} \\ E^{-\mathrm{T}} \end{pmatrix},$$

further let

$$\tilde{Q}\begin{pmatrix} s_1 \\ u \end{pmatrix} = \begin{pmatrix} \tilde{s}_1 \\ \tilde{u} \end{pmatrix},$$

hence $\tilde{w} = \tilde{R}^{-1}\tilde{s}_1$.

Note that $w = R^{-1}s_1$ and hence from the definition of $\tilde{V}$, we have the relation

$$\begin{aligned} \begin{pmatrix} u - Y^{\mathrm{T}}w \\ w \end{pmatrix} &= \begin{pmatrix} \tilde{V}^{\mathrm{T}}s_1 + u \\ R^{-1}s_1 \end{pmatrix} = \begin{pmatrix} \tilde{V}^{\mathrm{T}} & I \\ R^{-1} & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ u \end{pmatrix} \\ &= \begin{pmatrix} \tilde{V}^{\mathrm{T}} & I \\ R^{-1} & 0 \end{pmatrix}\tilde{Q}^{-1}\tilde{Q}\begin{pmatrix} s_1 \\ u \end{pmatrix} = \begin{pmatrix} 0 & \tilde{D}^{\mathrm{T}} \\ \tilde{R}^{-1} & E \end{pmatrix}\begin{pmatrix} \tilde{s}_1 \\ \tilde{u} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{D}^{\mathrm{T}}\tilde{u} \\ \tilde{R}^{-1}\tilde{s}_1 + E\tilde{u} \end{pmatrix} = \begin{pmatrix} \tilde{D}^{\mathrm{T}}\tilde{u} \\ \tilde{w} + E\tilde{u} \end{pmatrix}. \end{aligned} \tag{20}$$

Then we get

$$\begin{pmatrix} u - Y^{\mathrm{T}}w \\ w \end{pmatrix} = \begin{pmatrix} \tilde{D}^{\mathrm{T}}\tilde{u} \\ \tilde{w} + E\tilde{u} \end{pmatrix},$$

$$\tilde{w} = w - E\tilde{u} = w - E\tilde{D}^{-T}(u - Y^{T}w). \quad \square \tag{21}$$

This algorithm computes $\tilde{w}$, where $\tilde{w}$ solves Eq. (12).

Algorithm 3.
INPUT: An upper triangular matrix R^{-T} and w, where $QX = R$ and w solves (11). A new set of k observations $(Y^{T} \quad u)$ and $M = \mathrm{diag}(\mu_1, \ldots, \mu_m)$.
1. Compute $\tilde{V} = -R^{-T}Y$.
2. Find $\tilde{Q}^{-T} = Q_n^{T}Q_{n-1}^{T} \cdots Q_1^{T}$. where Q_i is a row M^{-1}-invariant reflection, such that

$$\tilde{Q}^{-T}\begin{pmatrix} \tilde{V} \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}.$$

3. Update R^{-T} to $\tilde{R}^{-T}$, i.e,

$$\tilde{Q}^{-T}\begin{pmatrix} R^{-T} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-T} \\ E^{T} \end{pmatrix}.$$

4. Update w to $\tilde{w}$, i.e,

$$\tilde{w} = w - E\tilde{D}^{-T}(u - Y^{T}w).$$

The cost in flops for each step is: 1. $kn^2/2$, 2. $15k^2n$, 3. kn^2, 4. $\frac{3}{2}k^2 + 2kn$ with a total cost for Algorithm 3 as $\frac{3}{2}kn^2 + 15k^2n + 2kn + \frac{3}{2}k^2$ flops. A straightforward implementation of the rank-1 method of Pan and Plemmons [4] would require $\frac{5}{2}kn^2 + O(kn)$ multiplications. Thus, roughly speaking, Algorithm 3 requires less flops when $n \geqslant 15k$.

4. Inverse downdating

Let the matrix X and the vector s be given by the partition

$$s = \begin{pmatrix} \tilde{s} \\ d \end{pmatrix}, \quad X = \begin{pmatrix} \tilde{X} \\ z^{T} \end{pmatrix},$$

where $z^{T} \in \mathbb{R}^{k\times n}, d \in \mathbb{R}^{k}$. Then the problem

$$\min \|M_3^{-1/2}(\tilde{s} - \tilde{X}\tilde{w})\|_2 \tag{22}$$

is our downdating problem. Thus, we assume that we have the solution to (11) where $M_1 = \mathrm{diag}(M_3, M_4)$ and want the the solution to (22) by our row hyperbolic M-invariant method.

Assume that $Q = Q_n \cdots Q_1$, where Q_i are hyperbolic M-invariant reflections. Then we define $\hat{Q} = Q_1 \cdots Q_n$ to be used in the sequel.

We have the following lemma that is used to construct the downdating algorithm.

Lemma 4.1. *Assume that* $R^{\mathrm{T}}M_3^{-1}R - zM_4^{-1}z^{\mathrm{T}} > 0$. *Let* $\tilde{V} = -R^{-\mathrm{T}}z$ *and* $Q, \hat{Q}$ *be row hyperbolic M-invariant matrices such that*

$$\hat{Q}^{\mathrm{T}}\begin{pmatrix} \tilde{V} \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}.$$

Further, assume that

$$Q\begin{pmatrix} R \\ z^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}$$

then

$$\hat{Q}^{\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-\mathrm{T}} \\ F^{\mathrm{T}} \end{pmatrix}.$$

Proof. We have a row hyperbolic M-invariant matrix Q, such that

$$Q\begin{pmatrix} R \\ z^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}.$$

Since

$$\begin{aligned} I &= \begin{pmatrix} R^{-1} & 0 \end{pmatrix}\Phi\begin{pmatrix} R \\ z^{\mathrm{T}} \end{pmatrix} \\ &= \begin{pmatrix} R^{-1} & 0 \end{pmatrix}\hat{Q}\Phi Q\begin{pmatrix} R \\ z^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} U^{-1} & F \end{pmatrix}\Phi\begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix} U^{-1} & F \end{pmatrix} = \begin{pmatrix} R^{-1} & 0 \end{pmatrix}\hat{Q}$ then

$$\hat{Q}^{\mathrm{T}}\begin{pmatrix} R^{-\mathrm{T}} \\ 0 \end{pmatrix} = \begin{pmatrix} U^{-\mathrm{T}} \\ F^{\mathrm{T}} \end{pmatrix}.$$

If U is upper triangular, then $U = \tilde{R}$. □

Theorem 4.2. *Assume that Q is hyperbolic M-invariant and $\hat{Q}$ satisfies*

$$\hat{Q}^{\mathrm{T}}\begin{pmatrix} \tilde{V} & R^{-\mathrm{T}} \\ I_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{R}^{-\mathrm{T}} \\ \tilde{D} & F^{\mathrm{T}} \end{pmatrix}. \tag{23}$$

If w is the solution to (11), *then the solution to* (22) *is given by*

$$\tilde{w} = w + F\tilde{D}^{-\mathrm{T}}(d - z^{\mathrm{T}}w). \tag{24}$$

Proof. The proof is analogous to that of Theorem 3.3 and is omitted. □

The following algorithm makes an inverse downdating.

Algorithm 4.

INPUT: The lower triangular matrix R^{-T} and w where $\tilde{Q}X = R$ and w solves Eq. (11). A set of k observations $(z^T \ \ d)$, and $M = \mathrm{diag}(\mu_1, \ldots, \mu_m)$. Then this algorithm computes $\tilde{w}$, where $\tilde{w}$ solves Eq. (22).

1. Compute $\tilde{V} = -R^{-T}z$.

2. Find $Q = Q_n Q_{n-1} \cdots Q_1$, where Q_i is row hyperbolic M-invariant reflection, such that

$$\hat{Q}^T \begin{pmatrix} \tilde{V} \\ I_k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{D} \end{pmatrix}.$$

3. Downdate R^{-T} to $\tilde{R}^{-T}$,

$$Q^T \begin{pmatrix} R^{-T} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{R}^{-T} \\ F^T \end{pmatrix}.$$

4. Downdate w to $\tilde{w}$,

$$\tilde{w} = w + F\tilde{D}^{-T}(d - z^T w).$$

The cost in flops is for each step: 1. $kn^2/2$, 2. $15k^2n$, 3. kn^2, 4. $\frac{3}{2}k^2 + 2kn$. The total cost is the same as for Algorithm 3, i.e., $\frac{3}{2}kn^2 + 15k^2n + 2kn + \frac{3}{2}k^2$ flops. The method of Pan and Plemmons [4] requires $\frac{5}{2}kn^2 + O(kn)$ multiplications.

5. Numerical experiments and remarks

In this section we provide some numerical experiments, In each of the examples given below, we indicate the length of the window used, and the number of observations which will be added or deleted.

The numerical tests for the examples were performed using Matlab, and the right-hand-side vector was chosen to be the row sums of the date matrix. Thus the exact solution is the vector of all ones. The quantites reported are the relative errors for our method and the methods of Pan and Plemmons [4] and Bojanczyk et al. [3].

Example 1. In this example we construct a 50×5 matrix whose entries are $1/(i+j-1), i = 1, \ldots, 50, \ j = 1, \ldots, 5$ where the exact solution is known (see Table 1). We use the Algorithm 1 and 3, and compare our methods to the rank-1 rotation-based methods of Pan and Plemmons [4]. An M indicates our methods and I indicates Pan and Plemmons' methods.

Example 2. In this example, we use our block methods, and the number of observations added and deleted is $k = 5$. In Table 2, we compare our method

Table 1
Relative error of rank-1 update

k	M	I
3	5.8×10^{-13}	4.9×10^{-12}
6	4.6×10^{-12}	9.0×10^{-12}
9	1.7×10^{-12}	3.8×10^{-12}
12	1.7×10^{-12}	1.8×10^{-12}
15	4.5×10^{-14}	1.6×10^{-12}
18	6.3×10^{-13}	2.3×10^{-12}
21	1.6×10^{-12}	5.4×10^{-12}
24	1.8×10^{-12}	1.1×10^{-12}
27	4.4×10^{-13}	2.1×10^{-13}
30	4.6×10^{-13}	2.3×10^{-12}
33	4.3×10^{-13}	3.2×10^{-12}
36	3.0×10^{-13}	1.9×10^{-12}
39	3.8×10^{-13}	1.8×10^{-12}
42	6.7×10^{-13}	1.4×10^{-12}
45	4.1×10^{-13}	1.2×10^{-13}

Table 2
Relative error of block method

k	M	I
1	3.067×10^{-12}	3.782×10^{-12}
2	1.859×10^{-12}	1.933×10^{-12}
3	1.267×10^{-12}	1.322×10^{-12}
4	7.554×10^{-13}	8.745×10^{-13}
5	6.650×10^{-13}	6.874×10^{-13}
6	3.032×10^{-14}	4.001×10^{-13}
7	1.645×10^{-13}	4.001×10^{-13}
8	8.404×10^{-14}	2.962×10^{-13}
9	8.404×10^{-14}	1.814×10^{-13}

and the BRLS method of Ref. [3]. M indicates our method and I indicates the BRLS method.

Example 3. In this example, we add a random number δ to all the entries in order to control the degree of ill-conditioning (see Table 3). The smaller the value of δ, the more ill conditioned is the matrix. We use $\delta = 10^{-5}$ and $\delta = 10^{-9}$. Here we again choose $k = 5$.

In Algorithm 3, we can know that the required flops of the step 2 are less. It is easy to see that computational cost in the row M-invariant reflection is not more than that in the orthogonal rotation method. Hence the method obtained in this paper is efficient. The above tables show that the error of our block

Table 3
Relative error

k	M (10×10^{-5})	I	M (10×10^{-9})	I
1	9.376×10^{-3}	3.436×10^{-2}	9.376×10^{-7}	3.436×10^{-6}
2	5.424×10^{-2}	7.587×10^{-2}	5.424×10^{-6}	7.587×10^{-6}
3	4.524×10^{-2}	5.264×10^{-2}	4.513×10^{-6}	5.264×10^{-6}
4	3.081×10^{-2}	2.668×10^{-2}	3.081×10^{-6}	2.668×10^{-6}
5	1.759×10^{-2}	8.072×10^{-2}	1.759×10^{-6}	8.072×10^{-7}
6	8.620×10^{-3}	2.370×10^{-2}	8.620×10^{-7}	2.370×10^{-6}
7	1.193×10^{-2}	4.436×10^{-2}	1.119×10^{-6}	4.436×10^{-6}
8	2.074×10^{-2}	6.419×10^{-2}	2.074×10^{-6}	6.419×10^{-6}
9	2.959×10^{-2}	8.307×10^{-2}	2.958×10^{-6}	8.307×10^{-6}

method is smaller. Our block method is better when the problem becomes ill-conditioned. From these small test samples, although it can not be concluded that the presented method of this paper is more stable or accurate, it is at least as good as the other.

As an example, we construct a 3×4 date matrix and continue with the updating and downdating process. Elements of this matrix are $1/(i+j)$, $i = 1, \dots, 4; j = 1, 2, 3$. Every element of the exact solution is 1. Taking $M = I$, then the computed solution is

$$x = (1.000006, 0.999980, 1.000016)^{\mathrm{T}}.$$

Let $M = \mathrm{diag}(0.5, 0.25, 0.1667, 1)$, then the computed solution is

$$x = (0.999999, 1.000003, 0.999997)^{\mathrm{T}}.$$

This example shows that the algorithm by selecting proper weight matrix would be efficient for the presented ill-conditioned problems. In general, the solution would be affected for chosen different weight matrix, but the choice of the best weight matrix is still an open problem.

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